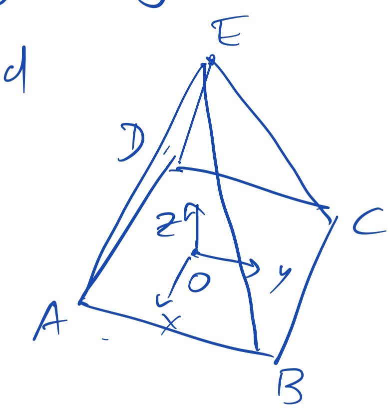
Answer model Symmetry in Physics exam of April 5, 2023

Exercise 1 pyramid



symmetry group C_{4v}

1a) rotation c around z over $90°$ $\quad (+0° = e)$

c^2 " $180°$

c^3 " $270°$.

reflection in xz plane b, in yz plane bc^2

" in BDE plane bc, in ACE plane bc^3.

(isomorphic to D_4, but subgp of $O(3)$, not $SO(3)$)

Conjugacy classes: (e), $(c) = \{c, c^3\}$, $(c^2) = \{c^2\}$

$(b) = \{b, bc^2\}$, $(bc) = \{bc, bc^3\}$

1b) 5 classes = 5 irreps. $\left.\begin{array}{l}\sum_{i=1}^{5} n_i^2 = 8 \\ n_1 = 1\end{array}\right\} \Rightarrow \begin{array}{l} n_2 = n_3 = n_4 = 1 \\ n_5 = 2\end{array}$

	(e)	(c)	(c^2)	(b)	(bc^2)
$D^{(1)}$	1	1	1	1	1
$D^{(2)}$	1	1	1	-1	-1
$D^{(3)}$	1	-1	1	1	1
$D^{(4)}$	1	-1	1	-1	-1
$D^{(5)}$	2	a	b	c	d

fill by homomorphism (rows $D^{(2)}$–$D^{(4)}$):

$\chi(c)^4 = 1 \Rightarrow \chi(c) \in \{\pm 1, \pm i\}$

$\chi(b)^2 = 1 \Rightarrow \chi(b) \in \{\pm 1\}$

$\underbrace{\chi(b)}_{\pm 1}\underbrace{\chi(c)}_{\pm 1} = \underbrace{\chi(bc)}_{\pm 1}$

fill by $\perp$ (row $D^{(5)}$):

$a = c = d = 0$, $b = -2$

1c) $\chi^V(\theta) = 1 + 2\cos\theta \Rightarrow \chi^V(e) = 3$

$\chi^V(c) = 1$

$\chi^V(c^2) = -1$

for b & bc one uses $D^V(b) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \Rightarrow \chi^V(b) = 1$

$$D^V(b)\, D^V(c) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \underbrace{\begin{pmatrix} c\theta & s\theta & \\ -s\theta & c\theta & \\ & & 1 \end{pmatrix}}_{\begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 1 \end{pmatrix}} \Rightarrow \chi^V(bc) = 1$$

$$\chi^V = (3, 1, -1, 1, 1)$$

$$\langle \chi^{(1)}, \chi^V \rangle = \frac{1}{8}(3 + 2.1 - 1 + 4.1) = 1$$

$\Rightarrow$ yes, allows for invariant vector.

Exercise 2

$O(3)$ acting on σ_{ij}

2a) $\sigma_{ij} = \delta_{ij} \xrightarrow{O(3)} D^V_{ki} \underbrace{D^V_{lj}}_{D^{V\,T}_{il}} \delta_{ij} = \delta_{kl}$ ← orthogonal matrices

2b) $\sigma'_{kl} = D^V_{ki} D^V_{lj} \sigma_{ij}$

If $\sigma'_{kl} = \sigma_{kl} \Rightarrow \quad \sigma_{kl} = D^V_{ki} D^V_{lj} \sigma_{ij}$

$$\Rightarrow \sigma_{kl} D^V_{lm} = D^V_{ki} \sigma_{ij} \underbrace{D^{V\,T}_{jl} D^V_{lm}}_{\delta_{jm}}$$

$$\Rightarrow \sigma_{kl} D^V_{lm} = D^V_{ki} \sigma_{im}$$

$$\Rightarrow [D^V, \sigma] = 0$$

2c) Determine all D^V that commute with

$$\sigma = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1+a \end{pmatrix} \Rightarrow D^V = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix}$$

demand orthogonality: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(2)$ & $e = \pm 1$

[In other words all rotations in the x-y plane
and all reflections in the x-y, x-z & y-z plane

description is sufficient

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

↳ and all their products: $\begin{pmatrix} R(\alpha) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \pm 1 \end{pmatrix}$, $\begin{pmatrix} PR(\alpha) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \pm 1 \end{pmatrix}$,

Ex 3 $SU(2)$, spin state $|s, m_s\rangle$, $U(\theta, \hat{n}) = \exp\left(\frac{i}{\hbar}\theta \hat{n}\cdot S\right)$

3a) S_z acting on $\begin{pmatrix} |\frac{1}{2} \; \frac{1}{2}\rangle \\ |\frac{1}{2} \; -\frac{1}{2}\rangle \end{pmatrix}$: $S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$U(\theta, \hat{n} = \hat{z}) = \exp\left(\frac{i}{\hbar}\theta S_z\right)$$

$$\Rightarrow D^{(\frac{1}{2})}(\theta, \hat{z}) = \exp\left(\frac{i}{\hbar}\theta \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$$

3b) If $\theta = 2\pi$, $D^{(\frac{1}{2})}(2\pi, \hat{z}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

hence $|\uparrow\rangle$ & $|\downarrow\rangle$ pick up a minus sign after
a 2π rotation around the $\hat{z}$ axis
(applies actually for all $\hat{n}$), Hence $D^{(\frac{1}{2})}(0) \neq D^{(\frac{1}{2})}(2\pi)$
as expected for $SO(3)$
so not rep of $SO(3)$

3c) $D^{(\frac{1}{2})} \otimes D^{(\frac{3}{2})} \sim D^{(1)} \oplus D^{(2)}$

$S_{tot} = \underbrace{|s_1 - s_2|}_{1}, \ldots, \underbrace{s_1 + s_2}_{2}$

$2S+1$

$\underline{2} \otimes \underline{4} \sim \underline{3} \oplus \underline{5}$

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